Lecture 08-09 - Ice Sheet Dynamics

The flow of ice in an ice sheet is driven by gravity. We call the process of gravity-driven flow a *gravity current*. We will introduce the steps required to obtain the "Shallow Ice Approximation". The analysis of a non-Newtonian ice sheet is slightly more involved than a Newtonian case. We first discuss a Newtonian gravity current, and then the non-Newtonian ice sheet dynamics. Here we consider a case with a zero bed slope $\theta = 0$.

Below is a list of parameters:

- ρ: fluid density, units: [kg/m^3].
- μ: (effective) fluid viscosity, units: [Pa·s]
- $B \approx 10^8$: ice hardness, units: [Pa·s$^{1/n}$]
- $\theta = \tan(\phi) \geq 0$: bed slope, units: dimensionless
- z: vertical coordinate, units: [m]
- W: width of the tank, units: [m]
- $-b(x) = -\theta x \leq 0$: bed elevation, units: [m]
- $H(x,t)$: fluid thickness, units: [m]
- $u(x,z,t)$: horizontal fluid velocity, units: [m/s]
- g: gravitational acceleration, units: [m/s^2]
- $n = 3$: Glen's flow law exponent, units: dimensionless
- $\phi \geq 0$: bed slope, units: [°]
- x : horizontal coordianbte, units: [m]
- t: time, units: [s]
- q_{in}: constant volumetric fluid flux per unit width supplied at $x = 0$, unit: [m^2/s]
- $h(x,t)$: surface height of the fluid with respective to the $z = 0$ plane, units: [m]
- $L(t)$: fluid front location, units: [m]
- $w(x,z,t)$: vertical fluid velocity, units: [m/s]

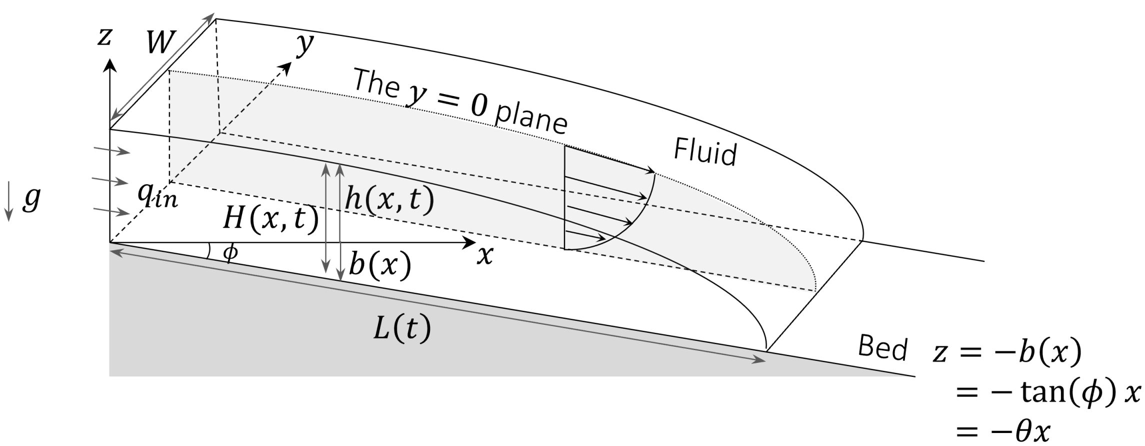

fig2 Figure 1: Schematic of a gravity current on a sloped bed. When the angle ϕ is small the fluid front location projected onto the x axis, $L(t)\cos(\phi)$, is approximately $L(t)$.

sec:1
1 Stokes equation

In the last lecture we introduced stress balances on a infinitesimally small fluid element and obtained the Stokes equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0, \tag{1}$$ stokes

where σ is the stress tensor and $\mathbf{f} = (f_x, f_y, f_z) = (0, 0, -\rho g)$ is the body force (unit: force per unit volume) acting on the fluid.

1.1 A simple 2D glacier

sec:1.1

Now we are going to solve the Stokes equation analytically by examining a simple 2D glacier stressb1 flow (a 2D *gravity current*).

$$x: \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \tag{2a}$$

$$z: \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g \tag{2b}$$

stress Recall that, according to the definition of deviatoric stress τ_{ij}

$$\sigma_{ij} = \tau_{ij} \quad \text{when} \quad i \neq j \tag{3a}$$

$$\sigma_{ij} = -p + \tau_{ij} \quad \text{when} \quad i = j, \tag{3b}$$

stressb2 where p is the fluid pressure. Equations (2) and (3) gives

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{4a}$$

$$z: \quad \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial p}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} = \rho g \tag{4b}$$

Next we assume all deviatoric normal stresses (τ_{xx}, τ_{zz}) are negligibly small compared to the stressb3 shear stress (τ_{xz}, τ_{zx}) (note that this is not true for ice shelves). Equation (4) become

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{5a}$$

$$z: \quad \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial p}{\partial z} = \rho g \tag{5b}$$

Recall that the for for a incompressible medium the deviatoric stress and strain rate follows the relationship

$$\tau_{ij} = 2\mu \dot{\epsilon}_{ij}, \quad \text{where} \quad \dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{6}$$ consti

According to the definition of strain rate $\dot{\epsilon}_{zx} = \dot{\epsilon}_{xz} = 1/2(\partial u/\partial z + \partial w/\partial x)$. Assuming all of the shear takes place in the plane normal to z, $\partial w/\partial x = 0$. The only non-zero component of the strain rate and the shear stress at the surface are

$$\dot{\epsilon}_{xz} = \dot{\epsilon}_{zx} \equiv \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\frac{\partial u}{\partial z} \quad \text{and} \tag{7a}$$
strainxz

$$\tau_{xz} = \tau_{zx} = 2\mu\dot{\epsilon}_{zx} = \mu\frac{\partial u}{\partial z}, \tag{7b}$$
stressxz

stressb4 respectively. Equations (5) and (7b) gives

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = 0 \tag{8a}$$

$$z: \quad \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial z}\right) - \frac{\partial p}{\partial z} = \rho g \tag{8b}$$

It can be demonstrated that $\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial z}\right)$ is much smaller compared to the body force ρg. Therefore Eqns (8) becomes

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = 0 \tag{9a}$$
stressb5

$$z: \quad -\frac{\partial p}{\partial z} = \rho g \tag{9b}$$
stressb6

Because the density of the air molecules is negligibly small compared to the density of ice, the fluid pressure at the surface of the current $z = h(x,t)$ is zero. Integrating equation (9b) vertically we obtain the fluid pressure p within the gravity current

$$p = \rho g(h - z) \quad \text{within} \quad 0 \leq z \leq h. \tag{10}$$
p

According to equation (9a), the fluid deformation is driven by the horizontal gradient of the fluid pressure. Substituting (10) into (9a) gives the x-direction of the Stokes equation

$$x: \quad -\rho g\frac{\partial h}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = 0 \quad \text{within} \quad 0 \leq z \leq h. \tag{11}$$
stressb7

This is the governing equation for a 2D gravity current, also called the "Shallow Ice Approximation" (SIA).

1.1.1 Boundary conditions

To solve for the velocity u in (11) we need two conditions for velocity u. Assume a no slip boundary condition at the bed.

$$u(z = 0) = 0. \tag{12}$$
ub

Note that if the bed is lubricated (like some places beneath ice sheets) this non-slip condition would not be true, but we will ignore that possibility here. In the corn syrup lab experiment, the non-slip condition at the bed holds. At the fluid surface the shear stress is zero (also called the free surface) because air exerts negligibly small shear stress on the fluid surface:

$$\tau_{zx}(z = h(x,t)) = 2\mu\dot{\epsilon}_{zx} = 0. \tag{13}$$
taus

According to the definition of strain rate $\dot{\epsilon}_{zx} = 1/2(\partial u/\partial z + \partial w/\partial x)$. Assuming all of the shear takes place in the plane normal to z, $\partial w/\partial x = 0$ and the zero shear stress condition at the surface (13) becomes

$$\tau_{zx}(z = h(x,t)) = \mu \frac{\partial u}{\partial z} = 0. \tag{14}$$ dudzs

However, the shear stress is non-zero within the fluid, which drives shear deformation of the fluid.

1.2 Lateral confinements

sec:1.2

So far we have been focusing on the velocity profile along an $x - z$ plane. In the experiment you will see interesting shear deformation at the fluid surface $z = h$. This is due to the effects from the lateral walls ($y = \pm W/2$). We can solve for the velocity profile at the fluid surface. We first extend the previous stress expression (4) to 3D:

3d1

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{15a}$$

$$y: \quad \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial p}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \tag{15b}$$

$$z: \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial p}{\partial z} = \rho g \tag{15c}$$

Note that there are many simplifications we can make when solving for the velocity at the fluid surface. First if $W \ll H$ the dominant shear stress gradient in (15) are $\partial \tau_{xy}/\partial x$ and $\partial \tau_{yx}/\partial y$. At the fluid surface, assuming all of the shear takes place in the plane normal to y, we have $\partial v/\partial x = 0$. The horizontal shear strain rate and the shear stress at the surface are

$$\dot{\epsilon}_{xy} = \dot{\epsilon}_{yx} \equiv \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\frac{\partial u}{\partial y} \quad \text{and} \tag{16a}$$ strainxy

$$\tau_{xy} = \tau_{yx} = 2\mu\dot{\epsilon}_{yx} = \mu\frac{\partial u}{\partial y}, \tag{16b}$$ stressxy

3d2 respectively. Substituting (16b) into (15),

$$x: \quad -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = 0 \tag{17a}$$ 3d2x

$$y: \quad \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial y} = 0 \tag{17b}$$

$$z: \quad -\frac{\partial p}{\partial z} = \rho g \tag{17c}$$ 3d2z

Similar to Section 1.1, integrating (17c) along z gives

$$p = \rho g(h - z) \quad \text{within} \quad 0 \le z \le h, \quad -W/2 \le y \le W/2. \tag{18}$$ p2

Substituting (18) into (17a), the velocity at the fluid surface obeys

$$x: \quad -\rho g \frac{\partial h}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \quad \text{within} \quad z = h(x), \quad -W/2 \leq y \leq W/2. \tag{19}$$

3dx2

Equation (19) looks very similar to (11)! In (11) the pressure gradient associated with $\partial h/\partial x$ is balanced by the vertical shear taking place in the plane normal to z. In (19) the pressure gradient associated with $\partial h/\partial x$ is balanced by the lateral shear taking place in the plane normal to y.

1.2.1 Boundary conditions

Because the system is symmetric with respect to the $y = 0$ plane, we can directly solve for $u(y)$ for the positive half plane $y > 0$ and the velocity at a particular $y < 0$ is the same as that at $y > 0$. To solve for u using (19) we needs two boundary conditions. The fluid at the walls obeys no-slip boundary condition:

$$u(x, z = h, y = W/2) = 0. \tag{20}$$

uw

At the $y = 0$ plane, a nonzero velocity gradient $\partial u/\partial y$ would result in a discontinuous velocity at $y = 0$. The second condition therefore requires

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad z = h. \tag{21}$$

sym

sec:2

2 A Newtonian gravity current

Consider a Newtonian fluid (constant viscosity μ) flowing down a rigid bed with $\theta = 0$. The schematics and the coordinate system are illustrated in figure 1. Fluid flux is supplied at a constant rate q_{in} at $x = 0$. The vertical velocity of the fluid is negligibly small compared with the horizontal velocity $w \ll u$. We consider the two cases described in Sections 1.1 and 1.2.

2.1 A simple 2D glacier

Let's first focus on the simple 2D glacier case (on the $x - z$ plane).

2.1.1 Velocity profile u

To solve (11) (a second order differential equation) for velocity u we can simply integration (11) along z twice and use two boundary conditions, (12) and (14), to get rid of the integration constants. Integrating (11) along z gives

$$-\rho g \frac{\partial h}{\partial x} z + \mu \frac{\partial u}{\partial z} + c_1 = 0, \tag{22}$$

u1

where c_1 is an integration constant. Integrating (22) along z again gives

$$-\frac{\rho g}{2}\frac{\partial h}{\partial x}z^2 + \mu u + c_1 z + c_2 = 0, \tag{23}$$

Applying the no-slip boundary condition (12) and the zero shear stress condition (14) to (23) and (22), respectively, we obtain

$$c_2 = 0 \tag{24a}$$

$$c_1 = \rho g h \frac{\partial h}{\partial x}. \tag{24b}$$

All together, (24) and (23) gives the velocity profile

$$u = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial x} z(2h - z) \quad \text{within} \quad 0 \le z \le h. \tag{25}$$

According to (25) the velocity at the fluid surface is

$$u_s = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial x} h^2. \tag{26}$$

Because the surface height gradient $\partial h/\partial x < 0$ is negative the fluid surface velocity $u_s > 0$ is positive. We expect fluid flowing in the positive x direction. We can rewrite the velocity (25) using (26):

$$u = u_s \frac{z}{h}(2 - \frac{z}{h}) \quad \text{within} \quad 0 \le z \le h. \tag{27}$$

2.1.2 Depth-integrated fluid flux q

Integrating the velocity (27), we obtain the horizontal fluid flux (volume per unit width W per unit time) crossing a x-plane is

$$q = \int_0^h u \mathrm{d}z = \frac{2}{3} u_s h. \tag{28}$$

2.1.3 Shear stress profile

The shear stress related to the shear strain rate via (assuming $\partial w/\partial x = 0$)

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}. \tag{29}$$

Substituting the velocity profile (25) into (44) we obtain that the shear stress

$$\tau_{zx} = -\rho g \frac{\partial h}{\partial x}(h - z) \quad \text{within} \quad 0 \le z \le h. \tag{30}$$

increases linearly with depth! $\tau_{zx} = 0$ at the fluid surface $z = h(x)$ (because when we solved for velocity we used the zero shear stress condition (14)) and $\tau_{zx} = -\rho g h \partial h/\partial x > 0$ at the fluid bed $z = 0$.

2.2 Lateral confinements

So far we have been focusing on the velocity profile along an $x - z$ plane. In the experiment you will see interesting shear deformation at the fluid surface $z = h$. Next let's examine the velocity profile at the fluid surface. The stress balance of fluid at the surface obeys (19). At the fluid surface, all of the shear takes place in the plane normal to y. Following similar steps as Section 1.1, equation (19) with boundary conditions ((20) and (21)) yields a parabolic surface velocity profile

$$u = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial x}((W/2)^2 - y^2) \quad \text{at} \quad z = h, \quad -W/2 \le y \le W/2. \tag{31}$$ laun1

sec:3
3 Ice sheet- a non-Newtonian gravity current

In this section we will extend the analysis for a Newtonian fluid to ice. Fluid flux is supplied at a constant rate q_{in} at $x = 0$. For simplicity let's consider $\theta = 0$ and $z = 0$ is at the bottom of the ice sheet. Ice is a shear-thinning fluid that follows Glen's flow law

$$\tau_{ij} = 2\mu\dot{\epsilon}_{ij}, \quad \text{where} \quad \mu = B/2\dot{\epsilon}_e^{1/n-1} \tag{32}$$ Glen

is the effectively ice viscosity, B is the ice hardness that is treated as a constant here, $n = 3$ is the exponent, and τ_{ij} and $\dot{\epsilon}_{ij}$ are the components in the stress and strain rate tensor, respectively. $\dot{\epsilon}_e$ is the effective strain rate.

3.1 A simple 2D glacier

For a simple 2D glacier, we assume all of the shear takes place in the plane normal to z. The only non-zero component in the stress and strain rate tensors are listed in (7a) and (7b). Thus $\dot{\epsilon}_e = \dot{\epsilon}_{zx}$ and the effective ice viscosity in eqn (32) becomes

$$\mu = \frac{B}{2}\dot{\epsilon}_e^{1/n-1} = \frac{B}{2}\dot{\epsilon}_{zx}^{1/n-1} = \frac{B}{2}\left(\frac{1}{2}\frac{\partial u}{\partial z}\right)^{1/n-1} \tag{33}$$ viscyz

The zero stress condition (14) can be written as:

$$\tau_{zx}(z = h(x,t)) = \mu\frac{\partial u}{\partial z} = B\left(\frac{1}{2}\frac{\partial u}{\partial z}\right)^{1/n} = 0. \tag{34}$$ zerostressice

3.1.1 Velocity profile u

To solve for velocity u, we substitute the effective viscosity of ice (33) into the Shallow Ice Approximation (11):

$$x: \quad -\rho g\frac{\partial h}{\partial x} + \frac{\partial}{\partial z}\left(B\left(\frac{1}{2}\frac{\partial u}{\partial z}\right)^{1/n}\right) = 0 \quad \text{within} \quad 0 \le z \le h. \tag{35}$$ ui1

Integrate (35) along z:

$$x: \quad -\rho g \frac{\partial h}{\partial x} z + B\left(\frac{1}{2}\frac{\partial u}{\partial z}\right)^{1/n} + c_1 = 0 \tag{36}$$ ui2

Apply the zero stress condition (34) to (36) we get $c_1 = \rho g h \partial h/\partial x$. Substitute c_1 into (36) gives

$$x: \quad -\rho g \frac{\partial h}{\partial x}(h - z) = B\left(\frac{1}{2}\frac{\partial u}{\partial z}\right)^{1/n} \tag{37}$$ ui3

We raise the both side of the equation to the n power.

$$x: \quad 2\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}(h - z)\right)^{n} = \frac{\partial u}{\partial z} \tag{38}$$ ui4

Integrate (38) along z

$$x: \quad -\frac{2}{n+1}\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}\right)^{n}(h - z)^{n+1} + c_2 = u \tag{39}$$ ui5

Apply the no slip condition to (39) gives $c_2 = \frac{2}{n+1}\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}\right)^n h^{n+1}$. Substitute c_2 into (39) gives the vertical velocity profile"

$$u = c_3(h^{n+1} - (h - z)^{n+1}), \quad \text{where} \quad c_3 = \frac{2}{n+1}\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}\right)^{n} \tag{40}$$ ui6

At the ice surface, (40) gives a surface velocity of

$$u_s = \frac{2}{n+1}\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}\right)^{n} h^{n+1} \tag{41}$$ uis

The vertical velocity profile (40) can be written in terms of u_s ((41)):

$$u = u_s\left(1^{n+1} - \left(1 - \frac{z}{h}\right)^{n+1}\right) \tag{42}$$ ui7

Using (41), $\rho \approx 10^3$ kg/m^3, $g \approx 10$ m/s^2, $h \approx 10^3$ m, $n = 3$, $B \approx 10^8$ Pa·s$^{1/3}$, and surface slope $-\partial h/\partial x \approx 0.02$, the surface velocity is about 200 m/yr. Note that the surface velocity is very sensitive to the choice of parameters for $\partial h/\partial x, B$, and h especially for large positive power exponent $2 \le n \le 4$.

3.1.2 Depth-integrated fluid flux q

Integrating the velocity (42), we obtain the horizontal fluid flux (volume per unit width W per unit time) crossing a x-plane is

$$q = \int_0^h u \mathrm{d}z = \frac{n+1}{n+2} u_s h. \tag{43}$$ qi

When $n \approx 4$ the flux is $q \approx 0.83 u_s h$ which is 24% higher than the flux $q \approx 0.67 u_s h$ estimated at $n \approx 1$.

3.1.3 Shear stress profile

The shear stress related to the shear strain rate via (assuming $\partial w/\partial x = 0$)

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}. \tag{44}$$ shear

Substituting the velocity profile (40) into (44) we obtain that the shear stress

$$\tau_{zx} = -\rho g \frac{\partial h}{\partial x}(h - z) \quad \text{within} \quad 0 \le z \le h. \tag{45}$$ shear2

is exactly the same as what we found for the Newtonian fluid! The shear stress increases linearly with depth, and independent of the effective viscosity of the fluid! The shear stress is the largest at the bed, $\tau_{zx} = -\rho g h \partial h/\partial x$ at $z = 0$.

3.2 Lateral confinements

Finally, let's examine the velocity profile at the fluid surface. At the fluid surface, all of the shear takes place in the plane normal to y. The nonzero component of strain rates are shown in (16a). The effective strain rate is therefore $\dot\epsilon_e = \dot\epsilon_{yx}$. The effective viscosity of ice (32) can be written as

$$\mu = \frac{B}{2}\dot\epsilon_e^{1/n-1} = \frac{B}{2}\dot\epsilon_{yx}^{1/n-1} = \frac{B}{2}\left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^{1/n-1} \tag{46}$$ viscyx

Substituting effective viscosity (46) into the governing equation (19), along with the boundary conditions (20) and (21), yields a surface velocity profile

$$u = \frac{2}{n+1}\left(-\frac{\rho g}{B}\frac{\partial h}{\partial x}\right)^n \left((W/2)^{n+1} - |y|^{n+1}\right) \quad \text{at} \quad z = h, \quad -W/2 \le y \le W/2. \tag{47}$$

When $n = 1$ this recovers (31) in which the viscosity $\mu = B/2$ would be a constant (independent of strain rate). For glacial ice $2 \le n \le 4$.